PARAMETER ESTIMATION FOR A CLASS OF DIFFUSION PROCESS FROM DISCRETE OBSERVATION

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Abstract. This paper is concerned with the parameter estimation problem for a class of diffusion process with drift coefficient $\alpha X_t^{2\gamma-1}$ and diffusion coefficient σX_t^{γ} from discrete observation. Euler-Maruyama scheme and iterative method are used to get the joint conditional probability density function. The maximum likelihood approach is applied for obtaining the parameter estimators and the explicit expressions of the error of estimation are given. The strong consistency of the estimators and asymptotic normality of the error of estimation are proved by using the law of large numbers for martingales, the strong law of large numbers and central-limit theorem. Hypothesis testing is made to verify the effectiveness of the estimation method used in this paper.

Keywords: Diffusion process, discrete observation, parameter estimation, strong consistency, asymptotic normality.

1. Introduction

Each field is more or less influenced by the random factors and diffusion process is an important tool to study the random phenomenon. Moreover, diffusion processes defined by stochastic differential equation are widely used for model building in astronomy, engineering, medical science and physical ([3, 13]). A recent application is in the area of financial economics ([5, 21]). The Black-Scholes option pricing model described by a geometric Brownian motion ([6]) and the Vasicek and Cox-Ingersoll-Ross models developed based on two specific mean-reverting diffusion processes ([22, 8, 9]) are widely used models in economic cases. However, in engineering practice, duo to the interference of random factors, part or all of parameters in diffusion process are always unknown. Parameters are needed to be estimated for the purpose of obtaining proper structures. Therefore, statistical inference for diffusion processes is of great importance from the theoretical as well as from an application point of view in model building.

In the past few decades, some methods have been used to estimate the parameters for diffusion process from continuous-time observation. For example, Kutoyants[16] used Bayes method to study the parameter estimation problem for diffusion process defined by nonlinear stochastic differential equation.

Yoshida[24] applied M-estimation to discuss the consistency of the parameter estimators. Khasminskii[14] considered the effectiveness of the estimators by using likelihood ratio function. Barczy[2] analyzed the consistency of the estimator by applying maximum likelihood estimation. Wei[23] used maximum likelihood approach to study the strong consistency of the estimator and asymptotic normality of the error of estimation. However, in fact, it is impossible to observe a process continuously in time. Therefore, parametric inference based on sampled data is important in dealing with practical problems. In earlier literatures, some methods have been applied to research the parameter estimation problem for continuous-time diffusion process from discrete observation. Prakasa Rao[20] used least squares estimation to study the consistency of the estimator. Florens-Zmirou[10] considered the weak convergence of the minimum contrast estimator. Bibby[4] constructed the martingale estimating function with zero mean to estimate the parameter for ergodic diffusion process. Jacod[12] discussed the convergence in probability of the estimator by using the construct function. Kuang[15] studied the Berry-Esseen boundedness of the estimator. Other methods such as generalized method of moments ([11]), Bayesian estimation([18, 19]) and approximation of the transition function([1, 7, 17]) have been used to estimate the parameters for diffusion processes as well.

In this paper, the parameter estimation problem for a class of diffusion process defined by a nonlinear stochastic differential equation is studied from discrete observation. Although parameter estimation for diffusion process has been investigated by many authors from discrete observation, the asymptotic normality of the estimator for the parameter in diffusion item and the hypothesis testing have not been discussed in earlier literatures. In our work, Euler-Maruyama scheme is used to discrete the process and the joint conditional probability density function is given. The explicit expression of the parameter estimators and the error of estimation are obtained. The strong consistency of the estimators and asymptotic normality of the error of estimation are proved by using the law of large numbers for martingales, the strong law of large numbers and central-limit theorem. Hypothesis testing is made to verify the effectiveness of the estimation method.

This paper is organized as follows. In Section 2, the joint conditional probability density function and the explicit expression of the parameter estimators are provided. In Section 3, the strong consistency of the estimators and asymptotic normality of the error of estimation are proved. In Section 4, hypothesis testing is made to verify the effectiveness of the estimators. Conclusion is given in Section 5.

2. Problem formulation and preliminaries

In this paper, we study the parameter estimation problem for a class of diffusion process described by the following nonlinear stochastic differential equation:

$$\begin{cases} dX_t = \alpha X_t^{2\gamma-1}dt + \sigma X_t^{\gamma} dB_t \\ X_0 = x_0. \end{cases} \tag{1}$$

where B_t is a standard Wiener process, α and σ are two unknown parameters, γ is a constant and $\gamma \in (1, \frac{3}{2}]$.

When $\gamma = 1$, (1) is a popular economic model called Black-Scholes Model. As it is a linear model, we do not consider it here. Due to the complexity of the transitional density function, it is difficult to obtain the commonly used expression for the unknown parameters. Therefore, numerical method should be used to obtain the approximate likelihood function.

From now on we shall work under the assumptions below.

Assumption 1. $\alpha < 0$*,* $\sigma > 0$*.* x_0 *is positive and independent with* B_t*.*

Assumption 2. $\sup_t \mathbb{E}|X_t| < \infty, \quad \sup_t \mathbb{E}\frac{1}{|X_t|} < \infty.$

Now the specific steps for obtaining the approximate likelihood function and the estimators is given below.

Let $Y_t = X_t^{1-\gamma}$, then equation (1) is changed to an equivalent equation, which is:

$$dY_t = (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_t}dt + (1-\gamma)\sigma dB_t. \tag{2}$$

It is assumed that the process is observed at times $\{t_0, t_1, ..., t_n\}$ where $t_i = i\Delta, i = 1, 2, ..., n$ and $\Delta > 0$. Discretizing equation (1), it follows that

$$Y_{t_i} - Y_{t_{i-1}} = (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta + (1-\gamma)\sigma\sqrt{\Delta}\varepsilon_{t_i}, \tag{3}$$

where $t_i = i\Delta$, ε_{t_i} is a i.i.d. N(0,1) sequence and for every i, ε_{t_i} is independent with $\{Y_{t_j}, j < i\}$.

Let $\mathcal{F}_{i-1} = \sigma(Y_{t_j}, j \le i-1)$. For the given $\mathcal{F}_{i-1}$, the conditional probability density function of Y_{t_i} is:

$$\begin{aligned} f(Y_{t_i}|\mathcal{F}_{i-1}) = & \frac{1}{\sqrt{2\pi\Delta}(1-\gamma)\sigma} \\ & \cdot \exp\{-\frac{(Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta)^2}{2(1-\gamma)^2\sigma^2\Delta}\}. \end{aligned} \tag{4}$$

Thus, for the given $\mathcal{F}_0$, the joint conditional probability density function of $\{Y_{t_1}, Y_{t_2}, ..., Y_{t_n}\}$ is:

$$\begin{aligned} f(Y_{t_1}, Y_{t_2}, ..., Y_{t_n}|\mathcal{F}_0) = & (\frac{1}{\sqrt{2\pi\Delta}(1-\gamma)\sigma})^n \cdot \\ & \prod_{i=1}^{n} \exp\{-\frac{(Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta)^2}{2(1-\gamma)^2\sigma^2\Delta}\}. \end{aligned} \tag{5}$$

Therefore, the likelihood function is given as follows:

$$L_n(\alpha,\sigma^2) = -\frac{n}{2}\ln\sigma^2 - \frac{1}{2(1-\gamma)^2\sigma^2\Delta} \cdot \sum_{i=1}^{n}(Y_{t_i} - Y_{t_{i-1}} - (1-\gamma)(\alpha - \frac{1}{2}\gamma\sigma^2)\frac{1}{Y_{t_{i-1}}}\Delta)^2. \tag{6}$$

Solving the equation set:

$$\begin{cases} \dfrac{\partial L_n(\alpha,\sigma^2)}{\partial\alpha} = 0 \\ \dfrac{\partial L_n(\alpha,\sigma^2)}{\partial\sigma^2} = 0, \end{cases} \tag{7}$$

we obtain the estimators:

$$\begin{cases} \widehat{\sigma^2} = \dfrac{\sum_{i=1}^{n}(Y_{t_i} - Y_{t_{i-1}})^2 \sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2} - (\sum_{i=1}^{n}\frac{Y_{t_i}-Y_{t_{i-1}}}{Y_{t_{i-1}}})^2}{n\Delta(1-\gamma)^2\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}} \\ \widehat{\alpha} = \dfrac{\gamma}{2}\widehat{\sigma^2} + \dfrac{\sum_{i=1}^{n}\frac{Y_{t_i}-Y_{t_{i-1}}}{Y_{t_{i-1}}}}{\Delta(1-\gamma)\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}. \end{cases} \tag{8}$$

3. Main results and proofs

In the following theorem, the strong consistency of two parameter estimators are proved by using the law of large numbers for martingales and the strong law of large numbers.

Theorem 1. *Under the Assumptions* (1) *and* (2), $\widehat{\sigma^2}$ *and* $\widehat{\alpha}$ *are strongly consistent.*

Proof. From (3), one has

$$\begin{aligned} &\sum_{i=1}^{n}(Y_{t_i} - Y_{t_{i-1}})^2 \sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2} \\ &= (1-\gamma)^2(\alpha - \frac{1}{2}\gamma\sigma^2)^2\Delta^2(\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2})^2 + (1-\gamma)^2\sigma^2\Delta\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}\sum_{i=1}^{n}\varepsilon_{t_i}^2 \\ &+2(1-\gamma)^2\sigma(\alpha - \frac{1}{2}\gamma\sigma^2)\Delta^{\frac{3}{2}}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}, \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&(\sum_{i=1}^{n}\frac{Y_{t_i}-Y_{t_{i-1}}}{Y_{t_{i-1}}})^2\\
&=(1-\gamma)^2(\alpha-\frac{1}{2}\gamma\sigma^2)^2\Delta^2(\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2})^2+(1-\gamma)^2\sigma^2\Delta(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2\\
&+2(1-\gamma)^2\sigma(\alpha-\frac{1}{2}\gamma\sigma^2)\Delta^{\frac{3}{2}}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}.
\end{aligned}
\tag{10}
$$

Substituting (9) and (10) into the expression of $\widehat{\sigma^2}$, it is checked that

$$
\widehat{\sigma^2}=\frac{\sigma^2\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}\sum_{i=1}^{n}\varepsilon_{t_i}^2-\sigma^2(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2}{n\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}. \tag{11}
$$

Thus, the error of estimation is

$$
\widehat{\sigma^2}-\sigma^2=\sigma^2(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{t_i}^2-1)-\frac{\sigma^2(\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}. \tag{12}
$$

Since ε_{t_i} is a i.i.d. N(0,1) sequence, $\varepsilon_{t_i}^2$ is also a i.i.d. sequence and $\mathbb{E}[\varepsilon_{t_i}^2]=1$. According to the strong law of large numbers, one has

$$
\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{t_i}^2-1\overset{a.s.}{\to}0. \tag{13}
$$

Now we will prove that $\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}$ is a martingale with zero mean with respect to the σ-algebra $\mathcal{F}_{n-1}=\sigma(\frac{1}{Y_{t_j}},\varepsilon_{t_j};0\le j\le n-1)$.

Since

$$
\mathbb{E}[\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}/\mathcal{F}_{n-1}]=\sum_{i=1}^{n-1}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}+\mathbb{E}[\frac{\varepsilon_{t_n}}{Y_{t_{n-1}}}/\mathcal{F}_{n-1}]=\sum_{i=1}^{n-1}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}},
$$

and

$$
\mathbb{E}[\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}]=\sum_{i=1}^{n}\mathbb{E}[\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}]=0,
$$

it follows that $\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}$ is a martingale with zero mean with respect to the σ-algebra $\mathcal{F}_{n-1}=\sigma(\frac{1}{Y_{t_j}},\varepsilon_{t_j};0\le j\le n-1)$. As $\mathbb{E}[(\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2]=\mathbb{E}[\frac{1}{Y_{t_{i-1}}^2}]$ is bounded, form the law of large numbers for martingales, we obtain that

$$
\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}\overset{a.s.}{\to}0,(n\to\infty). \tag{14}
$$

Thus, it can be checked that

$$(\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2 \overset{a.s.}{\to} 0, (n\to\infty). \tag{15}$$

Let

$$Y_M = \sup_{0\le t_{i-1}<\infty}\{Y_{t_{i-1}}\}, \quad Y_N = \inf_{0\le t_{i-1}<\infty}\{Y_{t_{i-1}}\}, \tag{16}$$

then we obtain that

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}} \le Y_M^2. \tag{17}$$

From above results, it is checked that

$$\widehat{\sigma^2} - \sigma^2 \overset{a.s.}{\to} 0, (n\to\infty). \tag{18}$$

Next we will prove that $\widehat{\alpha} - \alpha \overset{a.s.}{\to} 0, (n\to\infty)$.
From (3), one has

$$\sum_{i=1}^{n}\frac{Y_{t_i}-Y_{t_{i-1}}}{Y_{t_{i-1}}} = (1-\gamma)(\alpha-\frac{1}{2}\gamma\sigma^2)\Delta\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2} + (1-\gamma)\sigma\sqrt{\Delta}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}. \tag{19}$$

Sunstituting (19) into the expression of $\widehat{\alpha}$, we obtain that

$$\begin{aligned}\widehat{\alpha} &= \frac{\gamma}{2}\widehat{\sigma^2} + \frac{(1-\gamma)(\alpha-\frac{1}{2}\gamma\sigma^2)\Delta\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2} + (1-\gamma)\sigma\sqrt{\Delta}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{(1-\gamma)\Delta\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}\\ &= \frac{\gamma}{2}\widehat{\sigma^2} + \alpha - \frac{\gamma}{2}\sigma^2 + \frac{\sigma\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{\sqrt{\Delta}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}.\end{aligned}$$

Therefore, it can be checked that

$$\widehat{\alpha} - \alpha = \frac{\gamma}{2}(\widehat{\sigma^2} - \sigma^2) + \frac{\sigma\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}}}{\sqrt{\Delta}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}. \tag{20}$$

Since $\frac{1}{n}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}} \overset{a.s.}{\to} 0$ and $\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}} \le Y_M^2$, one has

$$\widehat{\alpha} - \alpha \overset{a.s.}{\to} 0, (n\to\infty). \tag{21}$$

Thus, $\widehat{\sigma^2}$ and $\widehat{\alpha}$ are strongly consistent.
The proof is complete. □

In the following theorem, the asymptotic normality of the error of estimation is proved by applying the law of large numbers for martingales, the strong law of large numbers and central-limit theorem.

Theorem 2. *Under the Assumptions* (1) *and* (2), $\sqrt{n}(\widehat{\sigma^2}-\sigma^2) \xrightarrow{d} N(0, 2\sigma^4)$.

Proof. From the expression of $\widehat{\sigma^2}-\sigma^2$, one has

$$\sqrt{n}(\widehat{\sigma^2}-\sigma^2) = \sigma^2\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{t_i}^2 - 1) - \frac{\sigma^2\sqrt{n}\frac{1}{n^2}(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}}. \tag{22}$$

Since $\mathbb{E}[\frac{1}{n}\sum_{i=1}^{n}(\varepsilon_{t_i}^2-1)] = 0$ and $var[\frac{1}{n}\sum_{i=1}^{n}(\varepsilon_{t_i}^2-1)] = \frac{2}{n}$, from the central-limit theorem, it is checked that

$$\sigma^2\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_{t_i}^2 - 1) \xrightarrow{d} N(0, 2\sigma^4). \tag{23}$$

Since

$$\mathbb{E}[\sqrt{n}\frac{1}{n^2}(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2] = \mathbb{E}[\sqrt{n}\frac{1}{n^2}\sum_{i=1}^{n}\frac{\varepsilon_{t_i}^2}{Y_{t_{i-1}}^2} + \sqrt{n}\frac{1}{n^2}\sum_{i\neq j}^{n}\frac{\varepsilon_{t_i}\varepsilon_{t_j}}{Y_{t_{i-1}}Y_{t_{j-1}}}] \to 0.$$

According to Chebyshev inequality, one has

$$\sqrt{n}\frac{1}{n^2}(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2 \xrightarrow{P} 0. \tag{24}$$

Thus, we obtain that

$$\frac{\sqrt{n}\frac{1}{n^2}(\sum_{i=1}^{n}\frac{\varepsilon_{t_i}}{Y_{t_{i-1}}})^2}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{Y_{t_{i-1}}^2}} \xrightarrow{P} 0. \tag{25}$$

From above results, one has

$$\sqrt{n}(\widehat{\sigma^2}-\sigma^2) \xrightarrow{d} N(0, 2\sigma^4). \tag{26}$$

The proof is complete. □

4. Hypothesis testing

In this section, we will introduce an example in the case of $\gamma = \frac{3}{2}$ in (1), which is described by the following stochastic differential equation:

$$\begin{cases} dX_t = \alpha X_t^2 dt + \sigma X_t^{\frac{3}{2}} dB_t \\ X_0 = x_0. \end{cases} \tag{27}$$

This is a economic model called Constantinides-Ingersoll model. We make the hypothesis testing for this example below.

We consider the testing problem as follows:

$H_0 : \sigma^2 = \sigma_0^2, \quad H_1 : \sigma^2 \neq \sigma_0^2,$

and the limit distribution of the test statistic based on this estimation.

Define:

$$Z_n = \frac{\sqrt{n}}{\sqrt{2}\sigma^2}(\widehat{\sigma^2} - \sigma^2), \tag{28}$$

then

$$Z_n \xrightarrow{d} N(0,1). \tag{29}$$

As the significance level is 0.05, if $|Z_n| \geq Z_{0.975}$, we refuse the original hypothesis.

When the confidence level is 0.95, the confidence interval of σ^2 is

$$[\frac{\widehat{\sigma^2}}{1 + \frac{\sqrt{2}}{\sqrt{n}} Z_{0.975}}, \frac{\widehat{\sigma^2}}{1 - \frac{\sqrt{2}}{\sqrt{n}} Z_{0.975}}]. \tag{30}$$

We assume that $\{\varepsilon_{t_i}\} \sim N(0,1)$. For every given true value of the parameters-α and σ^2, the size of the sample is represented as "Size n" and given in the first column of the table. In Table 1, $\Delta = 1$, the size of the sample is increasing from 100 to 500. In Table 2, $\Delta = 0.01$, the size is increasing from 10000 to 50000. These two tables list the value of "$\alpha - MLE$", "$\sigma^2 - MLE$" and the absolute error of MLE(Maximum Likelihood Estimator). Table 1 and Table 2 illustrate that the absolute error of α and σ^2 depend on the size of given value of α and σ^2. But under the hypothesis of normal distribution, there is no obvious difference between estimators and true value, estimators-$\widehat{\alpha}$, $\widehat{\sigma^2}$ are good.

Table 1: MLE simulation results of α and σ^2 $\Delta = 1$

True		Aver		AE	
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	σ^2	α
(0.1,-0.1)	100	0.1116	-0.0928	0.0116	0.0072
	200	0.1059	-0.0975	0.0059	0.0025
	500	0.0998	-0.1010	0.0002	0.0010
(0.3,-0.2)	100	0.3208	-0.1799	0.0208	0.0201
	200	0.3165	-0.1894	0.0165	0.0106
	500	0.2992	-0.2032	0.0008	0.0032
(0.5,-0.4)	100	0.5206	-0.3836	0.0206	0.0164
	200	0.5138	-0.3892	0.0138	0.0108
	500	0.4990	-0.4039	0.0010	0.0039
(0.7,-0.6)	100	0.6832	-0.6186	0.0168	0.0186
	200	0.6884	-0.6108	0.0116	0.0108
	500	0.6993	-0.6027	0.0007	0.0027

Table 2: MLE simulation results of α and σ^2 $\Delta = 0.01$

True		Aver		AE	
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	σ^2	α
(0.1,-0.1)	10000	0.1006	-0.0944	0.0006	0.0056
	20000	0.0996	-0.0932	0.0004	0.0068
	50000	0.0999	-0.0957	0.0001	0.0043
(0.3,-0.2)	10000	0.3010	-0.1973	0.0010	0.0027
	20000	0.2995	-0.1984	0.0005	0.0016
	50000	0.2998	-0.1998	0.0002	0.0002
(0.5,-0.4)	10000	0.5012	-0.3978	0.0012	0.0022
	20000	0.4996	-0.3994	0.0004	0.0006
	50000	0.4998	-0.4010	0.0002	0.0010
(0.7,-0.6)	10000	0.7025	-0.6008	0.0025	0.0008
	20000	0.6986	-0.6026	0.0014	0.0026
	50000	0.6995	-0.6035	0.0005	0.0035

Next we give some simulation results of the confidence interval of σ^2 under 0.95 confidence level. In Table 3, we suppose that $\{\varepsilon_i\} \sim N(0,1)$. For every given true value of σ^2, let $\Delta = 0.1$, the size of the sample is increasing from 1000 to 10000. This table lists the value of $\widehat{\sigma^2}$ and in the last column of the table lists the confidence interval of σ^2. Table 3 illustrates that the length of the confidence interval is becoming small when the size of the sample is increasing.

Table 3: simulation results of confidence interval of σ^2 $\Delta = 0.1$

True		Aver		0.95
(σ^2, α)	Size n	$\widehat{\sigma^2}$	$\widehat{\alpha}$	
(0.3,-0.2)	1000	0.3054	-0.2036	[0.2786,0.3324]
	2000	0.3032	-0.2025	[0.2875,0.3246]
	5000	0.3019	-0.2018	[0.2904,0.3148]
	10000	0.3008	-0.2006	[0.2931,0.3092]
(0.5,-0.4)	1000	0.5062	-0.4053	[0.4651, 0.5548]
	2000	0.5043	-0.4046	[0.4792, 0.5423]
	5000	0.5028	-0.4031	[0.4845,0.5247]
	10000	0.5016	-0.4017	[0.4883,0.5162]
(0.7,-0.6)	1000	0.7068	-0.6061	[0.6513,0.7768]
	2000	0.7051	-0.6052	[0.6704,0.7592]
	5000	0.7036	-0.6033	[0.6784,0.7334]
	10000	0.7019	-0.6010	[0.6832, 0.7221]
(0.9,-0.8)	1000	0.9157	-0.8103	[0.8378,0.9988]
	2000	0.9112	-0.8066	[0.8623,0.9763]
	5000	0.9064	-0.8048	[0.8722,0.9434]
	10000	0.9030	-0.8007	[0.8786,0.9287]

5. Conclusion

The aim of this paper is to estimate the parameters for a class of nonlinear stochastic differential equation. The likelihood function has been given by using Euler method, the explicit expressions of estimators and the error of estimation have been obtained. The strong consistency of the estimators and asymptotic normality of the error of estimation have been proved by using the law of large numbers for martingales, the strong law of large numbers, Chebyshev inequality and central-limit theorem. Further topics will consider the parameter estimation for commonly nonlinear stochastic differential equation and stochastic differential equation driven by Lévy noises.

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